Quasi-ideals and Bi-ideals of a Right Ternary Near-ring

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Abstract— Right ternary near-rings (RTNR) are generalization of their binary counterpart. The authors in their earlier papers have defined right ternary near-rings, zero-symmetric right ternary near-rings and prime ideals. In this paper N- subgroups, quasi-ideals and bi-ideals of an RTNR are defined and their basic characteristics are studied. It is established that in a zero-symmetric RTNR every quasi-ideal is a bi-ideal and in a regular zero-symmetric RTNR, a bi-ideal is a right ternary subnear-ring. A characterization theorem for left regular zero-symmetric RTNR in terms of completely semi-prime left N-subgroups is given.

Index Terms— Right ternary subnear-ring, zero-symmetric RTNR, right, lateral, left ideals, completely semi-prime ideal.

1 Introduction

THE notion of ternary algebraic system was introduced by Lehmer [3] in 1932. Ternary semigroups [5, 4, 7], ternary semirings [6] are some of the algebraic structures which involve ternary product. To deal with the concept of near-rings using ternary product Warud Nakkhasen and Bundit Pibaljommee [11] have applied the notion of ternary semiring to define left ternary near- rings, ternary subnear-rings and their ideals. The authors [8, 9] have defined in their earlier works right ternary near-rings, zero-symmetric right ternary near-rings and prime ideals.

In 1987, Chelvam and Ganesan [2] introduced and generalized the notion of quasi-ideals of near-rings which was introduced by Yakabe [10] in 1983 to bi-ideals. The regular near-ring was introduced by Beidlemann [1]. In 1989, Yakabe characterized regular zero-symmetric near-rings without nonzero nilpotent elements in terms of quasi-ideals.

In this paper N- subgroups, quasi-ideals and bi-ideals of a binary right near-ring is generalized to right ternary near-ring using the concept of ternary semirings. It is established that every N-subgroup (ideal) is a quasi- ideal and every quasi-ideal is a bi-ideal in a zero-symmetric RTNR. It is also established that in a regular zero-symmetric RTNR, a bi-ideal is a right ternary subnear-ring.

Left regular zero-symmetric RTNR are characterized in terms of completely semi-prime left N-subgroups. A characterisation theorem for a regular , idempotent and distributive zero-symmetric RTNR is also given.

2 Preliminaries

In this section we give the basic definitions that are necessary for the following sections of this paper.

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Definition 2.1[5] Let N be a non-empty set and [ ] be an operation defined from N×N×N to N called a ternary operation. Then (N , [ ]) is a *ternary semigroup* if for every x,y,z,u,v ∈ N, [[xyz]uv] = [x[yzu]v] = [xy[zuv]] =[xyzuv].

Definition 2.2[5] Let A, B and C be non-empty subsets of a ternary semigroup N. Then [ABC] = {[abc] ∈ N | a∈A, b∈B, c∈C}.

Definition 2.3[8] Let N be a non-empty set together with a binary operation + and a ternary operation [ ]: N×N×N → N. Then (N, +, [ ]) is a *right ternary near-ring* (a right ternary near ring is written as RTNR) if

(RTNR-1) (N, +) is a group (not necessarily abelian).

(RTNR-2) (N, [ ]) is a ternary semigroup.

(RTNR-3) [(a + b) c d] = [a c d] + [b c d], for every a,b,c,d in N.

Similarly we can define *left ternary near-ring* and *lateral ternary near ring*.

Definition 2.4 [11] A non-empty subset S of a right ternary near-ring is called a right *ternary subnear-ring* (RTSNR) if (i) x – y ∈ S if x, y ∈ S (ii) [SSS] ⊆ S.

Definition 2.5[11] Let N and N ′ be any two right ternary near rings. Then a mapping h :N→N′ is called a *right ternary near ring homomorphism* if (i) h(x+ y) = h(x) + h(y) , (ii) h([x y z]) = [h(x) h(y) h(z)],for every x,y,z ∈ N.

Definition 2.6[8] Let N be a right ternary near-ring. Let J be a normal subgroup (N, +). Then J is called (i) a *right ideal* of N if [J N N] ⊆ J (ii) a *left ideal* if [t t ′ (t″ +i)] - [t t′ t″] ∈ J (iii) a *lateral ideal* if [t (t′ + i) t″] - [t t′ t″] ∈ J where t, t′, t″ ∈ N , i ∈ J.
J is an *ideal* of N if it is a right, lateral and left ideal of N.

Definition 2.7 [9] If N is an RTNR then

$(N_0)_R$ = {n∈N | [0 n n′] = 0, ∀ n′∈N},

$(N_0)_M$ = {n∈N | [n 0 n′] = 0, ∀ n′∈N},

$(N_0)_L$ = {n∈N | [n n′ 0] = 0, ∀ n′∈N} are called *right zero-symmetric part, lateral zero-symmetric part* and *left zero-symmetric part* of N respectively and

N_0 = {n ∈ N | [n 0 0] = 0} is the *zero-symmetric part* of N .
If N = N_0 then N is called a zero-symmetric RTNR.

Note 2.8 If N is an RTNR then
(i) $(N_0)_R$ = N and N_0⊆ N.
(ii) $(N_0)_M$ = N_0.
For if n∈ N_0 then [n00] = 0 and therefore [n0n'] = [n [000] n'] = [n0 [00n']] = [n00] = 0 and hence N_0 ⊆ $(N_0)_M$. Obviously $(N_0)_M$ ⊆ N_0. Thus $(N_0)_M$ = N_0
(iii) $(N_0)_L$ ⊆ N_0.

Definition 2.9 [9] An ideal J of N is a *completely semi-prime ideal* if x^3∈J⇒x∈J.

3 N-SUBGROUPS

In this section N-subgroups and trio- RTNR which is an RTNR in which each one-sided N-subgroup is an N-subgroup are defined and a trio-RTNR is established as a zero-symmetric RTNR.

Definition 3.1A non-empty subset H of N is called an *N- subgroup* of N if (i) H is a subgroup of (R, +) (ii) [NNH] ⊆ H (iii) [NHN] ⊆ H (iv) [HNN] ⊆ H.

If (i) and (ii) hold then H is called a *left* N-subgroup. If (i) and (iii) hold then H is called a *lateral* N-subgroup. If (i) and (iv) hold then H is called a *right* N-subgroup.
Obviously if H is an N-subgroup of N then [HHN] ⊆H, [NHH]⊆H, [HNH]⊆H and every N-subgroup is a right ternary subnear-ring.

Example 3. 2 Let N = {0, x, y, z}. Define + as in Table (i) and [ ] on N by [x y z] = (x.y).z for every x, y, z ∈ N where. is defined as in Table (ii). Then (N , + , [ ]) is a right ternary near-ring and {0, x} is an N- subgroup of N. {0,y} is only a left N-subgroup as [yxz] = x∉{0,y} and [zyx] = x∉{0,y}. Also {0, z} is not a right N-subgroup as [zxz] = x ∉ {0,z} and not a left or lateral N-subgroups as [xzz] = x ∉{0,z}.

+	0	x	y	z
0	0	x	y	z
x	x	0	z	y
y	y	z	0	x
z	z	y	x	0

Table (i)

.	0	x	y	z
0	0	0	0	0
x	0	0	0	x
y	0	x	y	y
z	0	x	y	z

Table (ii)

Definition 3.3 If N is an RTNR and Q is a subset of N then the subsets N*Q*N and N*N*Q of N are defined as
N*Q*N = {[x y + i z]- [x y z]∈N |x,y,z ∈ N and i ∈Q } and
N*N*Q = {[x y z+ i]- [x y z]∈N |x,y,z ∈ N and i ∈Q}.

Proposition 3.4 In a zero-symmetric right ternary near-ring , [NNQ] ⊆ N*N*Q and [NQN] ⊆ N*Q*N.
Proof: Let x,y ∈ N and z ∈ Q . Then [xyz] = [xyz] + 0 = [x y 0+z] – [x y 0],as N is zero-symmetric and obviously R.H.S is in N*N*Q and hence [NNQ] ⊆ N*N*Q. Similarly [NQN] ⊆N*Q*N.

Remark 3.5 In a zero-symmetric right ternary near-ring every right, lateral, left ideal is respectively a right, lateral and left N-subgroup of N.

Proposition 3.6 If N is an RTNR then $(N_0)_L$ is a right N-subgroup of N.
Proof: If H= $(N_0)_L$ then for x, y ∈ H, [x-y z 0] = [xz0] - [yz0] = 0 and hence x-y∈ H. Now let u ∈ [HNN].Then u = [h n n']. Consider [u r 0] = [[h n n'] r 0] = [h [n n'r] 0] = [h n''0] =0 as h∈ H. Thus u ∈ H and hence $(N_0)_L$ is a right N-subgroup of N.

Remark 3.7 Let N = {0, x, y, z} and + on N be defined as in Table (iii) and [ ] be defined as [xyz] = (x.y).z where. is defined as in Table (iv).Then (N,+,[ ]) is a right ternary near-ring.
It can be easily seen that $(N_0)_L$ ={0}, right N-subgroup and is *not* a left or lateral N-subgroup. Also $(N_0)_M$ is *not* an N-subgroup. Since $(N_0)_M$ ={0,y} as [yxz] = x ∉{0,y},[xyz]=x∉{0,y} and [xzy]=x∉{0,y}

+	0	x	y	z
0	0	x	y	z
x	x	0	z	y
y	y	z	0	x
z	z	y	x	0

Table (iii)

.	0	x	y	z
0	0	0	0	0
x	x	x	x	x
y	0	x	y	z
z	x	0	z	y

Table (iv)

Definition 3.8 An RTNR N is called a *trio- RTNR* if every one-sided N-subgroup is an N-subgroup of N.
If a left (resp.lateral,right) N-subgroup is an N-subgroup then N is called a *left* (resp.*lateral,right*) *trio-RTNR.*

Example 3.9 Let N ={0,x,y,z} and let + on N be defined as in Table (v) and [ ] on N by [x y z] = (x.y).z for every x, y, z ∈ N where . is defined as in Table (vi). Then (N, +, [ ]) is a trio-RTNR.

+	0	x	y	z
0	0	x	y	z
x	x	0	z	y
y	y	z	0	x
z	z	y	x	0

Table (v)

.	0	x	y	z
0	0	0	0	0
x	0	x	0	0
y	0	0	y	0
z	0	0	0	z

Table (vi)

In Example 3.2, N is *not* a trio-RTNR as {0,y} is only a left N-subgroup but is *not* an N-subgroup of N.

Lemma 3.10 Every trio- RTNR is a zero-symmetric RTNR.
Proof: Let N be a trio- RTNR. Since {0} is a right N-subgroup and N is a trio-RTNR, {0} is a left and lateral N-subgroup and hence [xy0] = 0 and [x0y] = 0, for every x, y∈N. Hence by Note 2.8, N is a zero-symmetric RTNR.

Remark 3.11 The converse of the above lemma in general is not true. For , in Example 3.2, N is a zero-symmetric RTNR but {0, y} is a left N-subgroup and is not a right, lateral N-subgroup of N.

Definition 3.12 If N is an RTNR then the set of all *distributive elements* of N is defined as N_d = {x∈N | [xy (n+n′)] = [xyn] + [xyn′] and [x (n+n′) y] = [xny] +[x n′y] for all n, n′, y∈N}.
If N = N_d then N is called a *distributive RTNR.*

Example 3.13 (i) Let N = {0,x,y,z} and + on N be defined as in Table(vii) and [ ] be defined as [xyz] = (x.y).z where . is defined as in Table (viii).Then (N, +, [ ]) is a distributive right ternary near-ring.

+	0	x	y	z
0	0	x	y	z
x	x	0	z	y
y	y	z	0	x
z	z	y	x	0

Table (vii)

.	0	x	y	z
0	0	0	0	0
x	0	x	0	x
y	0	0	0	0
z	0	x	0	x

Table (viii)

(ii) Let N ={0,x,y,z} and + on N be defined as in Table (ix) and [ ] on N by [x y z] = (x.y).z for every x, y, z ∈ N where . is defined as in Table (x). Then (N, +, [ ]) is *not* a distributive right ternary near-ring as [xx(x+y)] = [xxz] = 0 and [xxx] + [xxy] = x + 0 = x.

+	0	x	y	z
0	0	x	y	z
x	x	0	z	y
y	y	z	0	x
z	z	y	x	0

Table (ix)

.	0	x	y	z
0	0	0	0	0
x	0	x	0	0
y	0	0	y	0
z	0	0	0	z

Table (x)

Definition 3.14 An element x∈N is called an *idempotent element* if [xxx] = x.
If all the elements in N are idempotents then N is called an *idempotent RTNR.*

Example 3.15 In Example 3.13(ii) N is an idempotent RTNR but N in Example 3.13 (i) is *not* an idempotent RTNR as [yyy] ≠ y.

Theorem 3.16 Let N be an RTNR.Then
(i) If x,y ∈ N ,then [Nxy] is a left N-subgroup. In particular [Nxx] is a left N-subgroup of N.
(ii) If x∈N is distributive then [xyN] is a right N-subgroup of N. In particular, [xxN] is a right N-subgroup of N.
Proof: (i) Let x, y∈ N and [Nxy] = H.Let u, v ∈ H .Then u –v = [nxy] – [n′xy] =[(n-n′) xy] ∈ H. Now If z ∈ [NNH] then z = [n n′ h] = [n n′ [n″x y]] = [[n n′ n″] xy] ∈ H as [n n′ n″] ∈ N. It is obvious that [Nxx] in particular is a left N-subgroup of N.
(ii) Let x∈ N and [xyN] = H.Let u, v ∈ H .Then u –v = [xyn] – [xy n′] = [xy(n-n′)] ∈ H, as x is distributive. Now if y∈ [HNN] then z = [hnn′] = [[xy n″] n n′] = [xy [n″ n n′]] ∈ H as [n″ n n′]∈ N.Thus [xyN] is a right N-subgroup of N.
It is obvious that [xxN] in particular is a right N-subgroup of N.

4 Quasi-ideal

In this section quasi-ideals are defined and in a zero-symmetric RTNR, N-subgroups and ideals are established as quasi-ideals. It is also established that an additive subgroup of N is a quasi–ideal if it is the intersection of right, lateral and left N-subgroups as well as ideals in a zero-symmetric RTNR.

Definition 4.1 A subset Q of an RTNR N is called a *quasi-ideal* of *N* if the following conditions hold:
(i)(Q, +) is a subgroup of (N, +).
(ii) [QNN] ∩ ([NQN] + [N [NQN]N]) ∩ [NNQ] ∩ ((N*Q*N) +(N*(N*Q*N)*N) ∩ (N*N*Q)⊆ Q.
Example 4.2 Let N be as in Example 3.7.Then {0, x} is a quasi ideal.
In a zero-symmetric RTNR quasi-ideals are defined as follows.

Definition 4.3 An additive subgroup Q of a zero-symmetric RTNR N is called a *quasi-ideal* of N if [QNN]∩([NQN] +[N[NQN]N])∩[NNQ] ⊆ Q.

Example 4.4 Let N be as in Example 3.2.Then {0, x} and {o, y} are quasi-ideals of N.

Proposition 4.5 Every quasi-ideal of a zero-symmetric RTNR N is an RTSNR of N.
Proof: As Q is a quasi-ideal of N, [QNN]∩([NQN] +[N[NQN]N])∩[NNQ] ⊆ Q. Now [QQQ] ⊆ [QNN], [QQQ] = [QQQ]+{0} ⊆ [NQN]+ [N[NQN]N] and [QQQ] ⊆ [NNQ]. Hence[QQQ]⊆[QNN]∩([NQN]+[N[NQN]N])∩[NNQ] ⊆ Q.Thus Q is an RTSNR of N.

Remark 4.6 The converse of the above proposition is in general not true. For in Example 3.2, Q={0,z} is an RTSNR but is *not* a quasi-ideal of N as [QNN]∩([NQN] +[N[NQN]N])∩[NNQ] is N ≠ Q.

Lemma 4.7 Let N be a zero-symmetric RTNR.Then
(i) Every right (resp.lateral, left) N-subgroup of N is a quasi-ideal of N.
(ii) Every right (resp.lateral, left) ideal of N is a quasi-ideal of N.
Proof: (i) Let Q be a right N-subgroup of N. Then Q is an additive subgroup of N and [QNN] ⊆ Q. Now [QNN]∩([NQN] +[N[NQN]N])∩[NNQ] ⊆[QNN] ⊆ Q. Thus Q is a quasi-ideal of N. Similarly if Q is a lateral or a left N-subgroup of N then Q is a quasi-ideal of N.
(ii) In a zero-symmetric right ternary near-ring every right, lateral, left ideal is respectively a right, lateral and left N-subgroup of N and hence by (i), every right (resp.lateral, left) ideal of N is a quasi-ideal of N.

Remark 4.8 In general the converse of the above lemma is not true. For in Example 3.2, N is a zero-symmetric RTNR and {0, y} is a quasi-ideal of N but is not a right and lateral N-subgroups (ideals) of N.

Lemma 4.9 Let N be a zero-symmetric RTNR.Then if Q is a quasi-ideal of N and T is an RTSNR of N then Q ∩ T is a quasi-ideal of T.

Proof: As $Q \cap T$ is a subgroup of (N, +) and $Q \cap T \subseteq T$, $Q \cap T$ is a subgroup of (T, +). Also,

$[(Q\cap T)\, TT]\cap([T(Q\cap T)T]+[T[T(Q\cap T)T]T]) \cap [TT(Q\cap T)])$
$\subseteq [QTT]\cap([TQT]+[T[TQT]T]) \cap[TTQ]$
$\subseteq[QNN] \cap([NQN] + [N[NQN]N])\cap[NNQ] \subseteq Q$.

Again

$[(Q\cap T)\, TT] \cap ([T\, (Q\cap T)\, T] + [T[T(Q \cap T)T]T]) \cap [TT(Q \cap T)]$
$\subseteq [TTT] \cap ([TTT] + [T\, [TTT]\, T]) \cap [TTT] \subseteq T$. Therefore
$[Q\cap T)TT]\cap([T(Q\cap T)T]+[TT[(Q\cap T)T]T]\,) \cap [TT(Q \cap T)] \subseteq Q\cap T$.
Thus $Q \cap T$ is a quasi-ideal of T.

Lemma 4.10 Let N be a zero-symmetric RTNR.Then the intersection of two quasi-ideals of N is a quasi-ideal of N.

Proof: Let Q_1 and Q_2 be quasi-ideals of N.Then they are additive subgroups of N and hence $Q_1\cap Q_2$ is an additive subgroup of N. Now,

$[(Q_1\cap Q_2)\, NN]\cap([N\, (Q_1\cap Q_2)\, N]+[N[N(Q_1\cap Q_2)N]N])$
$\cap\, [NN\, (Q_1\cap Q_2)]$
$\subseteq[Q_1NN]\cap ([NQ_1N]+[N[NQ_1N]N]) \cap[NNQ_1]\subseteq Q_1$.

Again$[(Q_1\cap Q_2)NN]\cap ([N(Q_1\cap Q_2)N]+[N[N(Q_1\cap Q_2)N]N]\,) \cap$
$[NN(Q_1\cap Q_2)]$
$\subseteq [Q_2NN] \cap ([NQ_2N] + [N[NQ_2N]N]) \cap[NNQ_2] \subseteq Q_2$.

Therefore$[(Q_1\cap Q_2)NN]\cap([N(Q_1\cap Q_2)N]+$
$[N[N(Q_1\cap Q_2)N]N]) \cap [NN(Q_1\cap Q_2)]\subseteq Q_1\cap Q_2$ .Thus $Q_1\cap Q_2$ is a quasi-ideal of N.

Corollary 4.11 The intersection of arbitrary collection of quasi-ideals of a zero-symmetric RTNR N is a quasi-ideal of N.

Lemma 4.12 Let N be a zero-symmetric RTNR.Then

(i) An additive subgroup of N is a quasi –ideal if it is the intersection of right, lateral and left N-subgroups of N.

(ii) An additive subgroup of N is a quasi –ideal if it is the intersection of right, lateral and left ideals of N.

Proof: (i) Let Q be an additive subgroup of N. Let J, M and L be right, lateral and left N-subgroups of N respectively.Then by Lemma 4.7(i), J, M and L are quasi-ideals of N. Let $Q = J\cap M\cap L$ .Then Q is the intersection quasi-ideals and hence by Corollary 4.11, Q is a quasi-ideal of N.

(ii) In a zero-symmetric right ternary near-ring every right, lateral and left ideal is respectively a right ,lateral and left N-subgroup of N and hence by (i), an additive subgroup of N is a quasi –ideal if it is the intersection of right,lateral and left ideal of N.

Proposition 4.14 If x is an idempotent element of an RTNR then $[xxNxx] = [xxN] \cap [Nxx]$.

Proof: If x is an idempotent element of an RTNR then $[xxx] = x$. Now if $u\in [xxNxx]$ then $u = [xxnxx] = [xx\, [nxx]]\in[xxN]$.
Also $u = [xxnxx] = [[xxn]\, xx] \in [Nxx]$.Hence $u \in [xxN] \cap [Nxx]$.

Suppose $v \in [xxN] \cap [Nxx]$.Then $v = [xxn_1]$ and $v = [n_2xx]$.
Consider $[xxvxx] = [xx[xxn_1]xx] = [[xxx]\, [xn_1x]x] =[[xxn_1]xx] = [vxx]=[[n_2xx]xx] = [n_2[xxx]x] = [n_2xx] = v$. Hence $v \in [xxNxx]$.
Thus $[xxNxx] = [xxN] \cap [Nxx]$.

Theorem 4.15 I Let N be a zero-symmetric RTNR.Then

(i) [Nxx] is a quasi-ideal of N.

(ii) If x is a distributive element of N then [xxN] is a quasi-ideal of N.

(iii) If x is an idempotent and distributive element of an RTNR then [xxNxx] is a quasi-ideal of N.

Proof: (i) By Theorem 3.16(i), [Nxx] is a left N-subgroup of N . Since every left N-subgroup is a quasi-ideal, [Nxx] is a quasi-ideal of N.

(ii) By Theorem 3.16(ii), [xxN] is a right N-subgroup of N. Since every right N-subgroup is a quasi-ideal, [xxN] is a quasi-ideal of N.

(iii) By Proposition 4.14, $[xxNxx] = [xxN] \cap [Nxx]$ and therefore by (i) and (ii) and Lemma 4.10, [xxNxx] is a quasi-ideal of N.

5 Bi-ideals

In this section bi-ideals of an RTNR N are defined and some of their basic algebraic properties are studied. It is established that an idempotent bi-ideal of a bi-ideal of a zero-symmetric RTNR N is a bi-ideal of N.

Definition 5.1 An additive subgroup J of N is called a *bi-ideal* of N if $[J\, [NJN]\, J] \cap J^*(N^*J^*N)^*J \subseteq J$.

Example 5.2 Let N be as in Example 4.2 then {0,x} is a bi-ideal of N.

In a zero-symmetric RTNR bi-ideals are defined as follows.

Definition 5.3 An additive subgroup J of N is called a *bi-ideal* of N if $[J[NJN]J] \subseteq J$.

Example 5.4 Let N be as in Example 3.2 .Then {0,x} is a bi-ideal of N.But L= {o,z} is not a bi-ideal as [LNLNL] is N and is not a subset of L.

Lemma 5.5 Let N be a zero-symmetric RTNR.Then every quasi-ideal of N is a bi-ideal of N.

Proof: Let L be a quasi-ideal of N.Then L is a subgroup of N.Now $[L\, [NLN]\, L] \subseteq [L[NNN]L]\subseteq[LNL]\subseteq[LNN] \rightarrow(1)$.
Also $[L[NLN]L]=\{0\}+[L[NLN]L]\subseteq[NLN]+[N[NLN]N]$ and $[L[NLN]L] \subseteq[L[NNN]L] \subseteq[LNL]\subseteq[NNL]$.
Thus$[L[NNN]L]\subseteq[LNN]\cap([NLN] +[N[NLN]N]) \cap [NNL] \subseteq L$, as L is a quasi-ideal of N.Hence L is a bi-ideal of N.

Lemma 5.6 Let N be a zero-symmetric RTNR.Then

(i) Every right (resp. lateral, left) N-subgroup of N is a bi-ideal of N.

(ii) Every right (resp. lateral, left) ideal of N is a bi-ideal of N

Proof: (i) In a zero-symmetric right ternary near-ring by Lemma 4.7(i) every right, lateral and left N-subgroup of N is a quasi-ideal of N. Hence by Lemma 5.5, every right (resp. lateral, left) N-subgroup of N is a bi-ideal of N.

(ii) In a zero-symmetric right ternary near-ring every right, lateral, left ideal is respectively a right, lateral and left N-subgroup of N. By Lemma 4.7(ii), every right (resp.lateral, left) ideal of N is a quasi-ideal of N. Hence by Lemma 5.5, every right (resp.lateral, left) ideal of N is a bi-ideal of N.

Proposition 5.7 If L is a bi-ideal of a zero –symmetric RTNR N and S is an RTSNR of N then L ∩ S is a bi-ideal of S.
Proof: Since L is a bi-ideal and S is an RTSNR L and S are additive subgroups of N and hence an additive subgroup of S also. Now [(L∩S) S (L∩S) S(L∩S)] ⊆ [LSLSL]⊆[LNLNL],as L is a bi-ideal of N.
Also[(L∩S) S (L∩S)S(L∩S)] ⊆ [S[SSS]S]⊆[SSS],as S is an RTSNR of N .
Thus [(L∩S) S (L∩S) S (L∩S)] ⊆ L∩S.
Hence L∩S is a bi-ideal of S.

Theorem 5.8 Let N be a zero-symmetric RTNR.Then an idempotent bi-ideal of a bi-ideal of N is a bi-ideal of N.
Proof: Let J be an idempotent bi-ideal of a bi-ideal L of N. Then [JJJ] = J → (1) ,[J[LJL]J] ⊆ J→(2) and [L[NLN]L] ⊆ L →(3).
Now [J[NJN]N] = [[JJJ] [NJN] [JJJ]] = [J[J[J [NJN]J]J]J]
⊆[J[J[L [NLN]L]J]J] ⊆[J[JLJ]J]
=[J[JLJ][JJJ]] ⊆ [J[J[LJL]J]J]
⊆ [JJJ] = J. Hence J is a bi-ideal of N.

Proposition 5.9 Let N be a zero-symmetric RTNR. If K is an RTSNR of N and J, M, L are right, lateral, left N-subgroups of N such that [JML]⊆K ⊆J∩M∩L then K is a bi-ideal of N.
Proof: Let J,M,L are right, lateral, left N-subgroups of N respectively, such that
[JML]⊆K⊆J∩M∩L.Now,[K[NKN]K]⊆[(J∩M∩L)N(J∩M∩L)N(J ∩M∩L)]⊆ [J[NML]L] ⊆[JML] ⊆K. Hence K is a bi-ideal of N.

Theorem 5.10 Let N be a zero-symmetric RTNR. Then
(i) If x,y ∈ N and J is a bi-ideal then [Jxy] is a bi-ideal of N
(ii) If x is a distributive element of N then [xyJ] is a bi-ideal of N. In particular, [Jxx] and [xxJ] are bi-ideals of N.
Proof: (i) Let J be a bi-ideal of N .Then [JNJNJ]⊆J. If u, v ∈[Jxy] then u = [sxy] and v= [txy]. Now u-v =[sxy]-[txy] = [s-t x y] ∈[Jxy].Also [[Jxy]N[Jxy]N[Jxy]] = [J[xyN]J[xyN]Jxy]⊆
[[JNJNJ]xy]⊆ [Jxy].Thus [Jxy] a bi-ideal of N.
(ii) Let x be a distributive element of N. If u, v∈[xyJ] then u = [xys] and v= [xyt].Now u-v =[xys]-[xyt] = [xy(s-t)] ∈[xyJ] as x is a distributive element .
Also [[xyJ]N[xyJ] N[xyJ]]
⊆[xyJ[Nxy]J[Nxy]J]⊆[xy[JNJNJ]]⊆[xyJ].
Thus [xyJ] a bi-ideal of N.
It is obvious that [Jxx] and [xxJ] are in particular bi-ideals of N.

Corollary 5.11 Let N be a zero-symmetric RTNR. If x,y ∈ N, if x is a distributive element of N and J is a bi-ideal of N then [x[yJx]y] is a bi-ideal of N. In particular, [xxJxx] is a bi-ideal of N.
Proof: Let J be a bi-ideal of N .Then [JNJNJ] ⊆J. If u, v∈[x[yJx]y] then u =[x[ymx]y] and v =[x[ynx]y] .
Now u - v = [x[ymx]y] - [x[ynx]y]= [x([ymx] - [ynx])y]= [x[y(m-n)x]y]∈[x[yJx]y] . Now[[x[yJx]y]N[x[yJx]y]N[x[yJx]y]] ⊆[[xyJ][x[yNx]y]J[x[yNx]y][Jxy]]⊆[[xyJ][NJN][Jxy]] ⊆[x[y[J[NJN]J]x]y]⊆ [x[yJx]y].Thus [x[yJx]y] is a bi-ideal of N.
It is obvious that [xxJxx] in particular is a bi-ideal of N.

6 Regular RTNR

In this section a regular RTNR is defined and its properties are studied. It is established that in a regular zero-symmetric RTNR, a bi-ideal is a right ternary subnear-ring. Moreover left regular zero – symmetric RTNR are characterized in terms of completely semi-prime left N-subgroups.A characterization theorem for a regular, idempotent and distributive RTNR is also given. Every intra-regular RTNR is proved as a lateral trio-RTNR.
Throughout this section N is taken as a zero-symmetric RTNR.

Definition 6.1 An RTNR is *regular* if for every x∈N there exists u∈N such that x = [xux].

Note 6.2 A regular element x∈N may equivalently be defined as x = [xuxvx], for some u, v∈N

Example 6.3 (i) Let N be as in example 3.13(ii).Then (N, +, [ ]) is a regular right ternary near-ring.
(ii) Let N be as in example 3.13(i). Then (N, +,[ ]) is *not* a regular right ternary near-ring for, y∈ N cannot be written as [y0y] or [yxy] or [yyy] or [yzy].

Note 6.4 If N is regular then it is obvious that [NNN] = N.

Lemma 6.5: Let N be a regular RTNR.Then
(i)Every lateral N- subgroup L of N is a regular RTSNR of N.
(ii) Every lateral ideal of N is a regular RTSNR of N.
Proof: (i) Let L be a lateral N-subgroup of N then [NLN] ⊆L. If x∈Lthen x∈N and hence by Definition 6.1, there exists u∈N such that x = [xux]. Now x = [xux] = [[xux]ux] =[x [uxu] x] ⇒ x =[xvx] where v = [uxu]∈ L. Since x = [xvx] where v∈ L it follows that [LLL] = L and hence L is a regular RTSNR.
(ii) In a zero-symmetric RTNR N, every lateral ideal of N is a lateral N-subgroup of N and hence by (i) the proof follows.

Lemma 6.6 If L is an N-subgroup of a regular RTNR N then [LLL] = L
Proof: Since L is an N-subgroup [LLL] ⊆L.Now let x∈L.Since N is regular x= [xux], for some u ∈ N.Since L is a lateral N-subgroup and x = [[xux] ux] = [x [uxu] x]∈[LLL].Thus L⊆[LLL].Hence [LLL] = L.

Theorem 6.7 The homomorphic image of a regular RTNR is a regular RTNR.
Proof: Let φ: N → M be an onto RTNR homomorphism. Let N be a regular RTNR. Suppose y ∈ M. Since φ is onto there exists x ∈ N such that φ(x) = y⇒φ ([xux]) = y ⇒ [φ(x) φ (u) φ(x)] = y ⇒[y φ(u)y] = y. Since φ (u) ∈ M this implies that M is a regular RTNR.

Theorem 6.8 If N is a regular RTNR and J, M, L are right, lateral and left N-subgroup of N respectively then [JML] = J∩M∩L.
Proof: Since [JML] ⊆ [JNN] ⊆J, [JML] ⊆ [NMN] ⊆M, [JML] ⊆ [NNL] ⊆ L it follows that [JML] ⊆J∩M∩L.
Now let x∈J∩M∩L. Since N is regular for some u∈N, x= [xux].
Consider x= [xux] = [xu[xux]] = [xu[[xux]ux]] = [x[uxu][xux]]

= [[xux][uxu][xux]]∈[JML] as J,M,L are right ,lateral and left N-subgroup of N respectively .This implies that J∩M∩L⊆[JML].Hence [JML]= J∩M∩L.

Corollary 6.9 If N is a regular RTNR and J, L are right and left N-subgroups of N respectively then [JNL] = J∩L.
Proof: As J is a right N-subgroup [JNL] ⊆ [JNN] ⊆J and [JNL] ⊆ [NNL] ⊆L as L is a left N-subgroup of N. Hence [JNL] ⊆ J∩L.Now let x∈ J∩L .Then as N is regular x = [xux] for some u∈N. Now x= [xux] = [[xux] u x] ∈ [JNL]. This implies that J∩L⊆ [JNL].Hence [JNL] = J∩L.

Remark 6.10 In a zero-symmetric regular RTNR Theorem6.8 and Corollary 6.9 hold good for ideals also.

Theorem 6.11 In regular RTNR an additive subgroup L of N is a bi-ideal iff L = [LNL].
Proof: Let L be an additive subgroup of an RTNR and [LNL] =L. Then [L [NLN] L] = [LN [LNL]] = [LNL] = L. Hence L is a bi-ideal of N.
Conversely let [L [NLN] L] ⊆L.Since N is regular, L⊆ [LNL]. Now [LNL] ⊆ [[LNL] NL] = [L [NLN] L] ⊆L.Hence L = [LNL].

Theorem 6.12 In a regular RTNR every bi-ideal is an RTSNR.
Proof: If L is a bi-ideal of N then L is an additive subgroup of N and as N is regular L = [LNL]. Now [LLL] ⊆ [LNL] = L and hence L is an RTSNR.

Definition 6.13 An element x∈N is called a *right (resp.left) regular* if there exists u∈N such that [xxu] = x (resp. [uxx] = x). If all the elements of N are right (resp.left) regular then N is called a *right (resp.left) regular RTNR.*

Note 6.14 A right (resp. left) regular element may equivalently be defined as x = [[xxx] uv] (resp. x= [uv [xxx]]), for some u, v∈N.

Example 6.15 In Example 3.2 , 0 and y are both right and left regular elements of N but N is neither a left regular nor a right regular RTNR as x ≠ [0xx] or [xxx] or [yxx] or [zxx] and also x≠[xx0] or [xxx] or [xxy] or [xxz].
The following theorem characterizes a left regular zero-symmetric RTNR in terms of a completely semi-prime left N-subgroup of N.

Theorem 6.16 A zero-symmetric RTNR is left regular if and only if every left N-subgroup of N is a completely semi-prime.
Proof : Let N be a left regular zero-symmetric RTNR and L be any left N-subgroup of N .Let for x ∈ N , x^3 = [xxx]∈L. Since N is left regular there exists u∈N such that x = [uxx].Now x = [u [uxx] x] = [u u [xxx]] ∈ [NNL]⊆L , as L is a left N-subgroup of N .Thus L is a completely semi-prime ideal of N.
Conversely let every left N-subgroup of N be completely semi-prime.Since for any x∈N by Proposition 3.16, [Nxx] is a left N-subgroup it follows that [Nxx] is completely semi-prime. Now x^3 = [xxx] ∈ [Nxx].Since [Nxx] is completely semi-prime x ∈ [Nxx] and hence x = [uxx], for some u∈N. Since x is arbitrary it follows that N is left regular.

Remark 6.17 (i) If N is a distributive right regular zero-symmetric RTNR then it can be characterized in terms of right N-subgroups.
(ii) If N is an idempotent, distributive RTNR then [Nxx] and [xxN] are written as $\langle x\rangle_l$ and $\langle x\rangle_r$ respectively.
The following is a characterization theorem for regular, idempotent and distributive RTNR.

Theorem 6.18 If N is an idempotent, distributive RTNR then the following statements are equivalent:
(i) N is regular.
(ii) R ∩ L = [RNL] for every right N-subgroup R and every left N-subgroup L of N.
(iii) $\langle x\rangle_r \cap \langle y\rangle_l$ = [$\langle x\rangle_r$ N$\langle y\rangle_l$] for every x,y∈ N.
(iv) $\langle x\rangle_r \cap \langle x\rangle_l$ = [$\langle x\rangle_r$ N$\langle x\rangle_l$] for every x ∈ N.
Proof: (i) ⇒ (ii) : By Corollary 6.8 , if N is a regular RTNR and R, L are right and left N-subgroups of N respectively then [RNL] = R∩L.
(ii) ⇒ (iii): As $\langle x\rangle_r$ and $\langle y\rangle_l$ are left N-subgroup and right N-subgroups of N respectively, $\langle x\rangle_r \cap \langle y\rangle_l$ = [$\langle x\rangle_r$ N$\langle y\rangle_l$] for every x,y∈ N follows by (ii).
(iii) ⇒ (iv): By taking y as x in (iii), $\langle x\rangle_r \cap \langle x\rangle_l$ = [$\langle x\rangle_r$ N$\langle x\rangle_l$] for every x ∈ N.
(iv) ⇒ (i) : Suppose (iv) holds. Since x∈$\langle x\rangle_r \cap \langle x\rangle_l$ = [$\langle x\rangle_r$ N$\langle x\rangle_l$] and [$\langle x\rangle_r$ N$\langle x\rangle_l$] = [[xxN]N[Nxx]] = [xx[NNN]xx] ⊆ [x[xNx]x]⊆ [xNx] , x∈[xNx] which implies that x = [xux] for some u∈N. Hence N is regular.

Theorem 6.19 In a regular RTNR N, the following statements hold.
(i) J = [JNJ] for every bi-ideal J of N,
(ii) Q = [QNQ] for every quasi-ideal Q of N.
Proof: Suppose N is a regular RTNR and let x ∈J. Then there exists u ∈N such that x = [xux] ⇒ x∈ [JNJ] ⇒ J⊆[JNJ] .
Now let y ∈ [JNJ] then y = [jnj′] for some j,j′∈J.and n∈N.
Since N is regular j = [jtj] for some t ∈N.Thus y = [jnj′] =
[[jtj] n j′] ∈ [JNJNJ] ⊆ J, which implies [JNJ] ⊆J.Hence [JNJ] = J
(ii) Since every quasi-ideal of an RTNR N is a bi-ideal, by (ii), Q = [QNQ] for every quasi-ideal Q of N.

Definition 6.20. An RTNR N is called *intra- regular* if for each element x ∈ N, there exists elements u, v ∈ N such that
[u [xxx] v] = x.

Example 6.21 (i) Let N be as in Example 3.13(ii).Then (N, +, [ ]) is an intra-regular RTNR.
(ii) Let N be as in Example 3.13(i).Then (N, +,[ ]) is *not* an intra-regular RTNR. For y∈ N cannot be written as [0$y^3$0] or [0y^3x] or [0y^3y] or [0y^3z] or [x$y^3$0] or [xy^3x] or [xy^3y] or [xy^3z] or [y$y^3$0] or [yy^3x] or [yy^3y] or [yy^3z] or [z$y^3$0] or [zy^3x] or [zy^3y] or [zy^3z].

Lemma 6.22 If N is an intra-regular RTNR then for any left N-subgroup L, lateral N-subgroup M and right N-subgroup J of S, L ∩ M ∩ J = [LMJ].
Proof: Suppose that N is an intra-regular RTNR. Let J, M and L be a right N-subgroup, a lateral N-subgroup and a left N-

subgroup of N respectively.
Since $[JML] \subseteq [JNN] \subseteq J$, $[JML] \subseteq [NMN] \subseteq M$, $[JML] \subseteq [NNL] \subseteq L$, it follws that $[JML] \subseteq J \cap M \cap L$.
Now for $x \in J \cap M \cap L$, we have $x = [ux^3v]$, for some $u, v \in N$. This implies that $x = [ux^3v] = [u\,[ux^3v][ux^3v][ux^3v]v] = [[uux^3]\,[vux^3vu][x^3vv]] \in [LMJ]$. Thus $J \cap M \cap L \subseteq [LMJ]$.

In the following theorem it is established that every intra-regular RTNR is a lateral trio-RTNR.

Theorem 6.23. Let N be an intra-regular RTNR. Then
(i) A non-empty subset M of N is an N-subgroup of N if and only if M is a lateral N-subgroup of N.
(ii) A non-empty subset M of N is an ideal of N if and only if M is a lateral ideal of N.
Proof: (i) It is obvious that if M is an N-subgroup of N then M is a lateral N-subgroup of N.
Conversely, let M be a lateral N-subgroup of an intra- regular RTNR. Let $z \in M$ and $x, y \in N$. Then $z \in N$ and hence there exists elements $u, v \in N$ such that $z = [uz^3v]$.
Now $[xyz] = [xy\,[uz^3v]] = [[xyu]\,z^3v] \in [NMN] \subseteq M$.
Also $[zxy] = [[uz^3v]\,xy] = [uz^3\,[vxy]] \in [NMN] \subseteq M$.
Thus M is an N-subgroup of N.
ii) In a zero-symmetric RTNR N, every lateral ideal of N is a lateral N-subgroup of N and hence by (i) the proof follows.

Lemma 6.24 Let N be an intra-regular RTNR. Then
(i) Every lateral N- subgroup L of N is an intra-regular RTSNR of N.
(ii) Every lateral ideal of N is an intra-regular RTSNR of N.
Proof: (i) Let L be a lateral N-subgroup of N then $[NLN] \subseteq L$.Let $x \in L$.Then $x \in N$. By Definition 6.20, there exists $u,v \in N$ such that $x = [ux^3v]$.Now $x = [ux^3v] = [u[ux^3v]\,[ux^3v]\,[ux^3v]v] = [[uux^3vu]x^3[vux^3vv]] \Rightarrow x = [u'x^3v']$ where $u' = [uux^3vu]$, $v' = [vux^3vv] \in L$.Since $x = [u'x^3v']$, $u',v' \in L$, L is regular which in turn implies that $[LLL] = L$ and hence L is a regular RTSNR.
(ii) In a zero-symmetric RTNR N, every lateral ideal of N is a lateral N-subgroup of N and hence by (i) every lateral ideal of N is an intra-regular RTSNR of N.

7 Conclusion

In this paper N-subgroups, quasi-ideals, bi-ideals and regular zero-symmetric right ternary near-rings were defined and their basic algebraic properties were proved. In a zero-symmetric right ternary near-ring, N-subgroups as well as ideals were established as quasi-ideals. Bi-ideals were visualized in the generalized context of quasi-ideals. In this work bi-ideals in a regular right ternary near-ring have been realized as right ternary subnear-rings. For future work a similar approach can be explored for right ternary near-rings which are not zero-symmetric.

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